Toward Vaught's Conjecture via Modeloids.

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Introduction:

In 1959 R. Vaught conjectured (see [1]) that a first order theory in a countable language has either continuum or at most countably many countable models. Sixty years later we still do not know whether this is true.

In this paper we show that for finite languages it is enough to prove Vaught's Conjecture (VC) for modeloids, certain equivalence relations:

- If Vaught's Conjecture is true for theories of modeloids then it is true in general for first order theories in a finite language.

The conjecture relates elementary equivalence of modeloids to their isomorphism classes. The results of the paper along these lines are:

- An algebraic representation of elementary equivalence of modeloids, $E_1 \equiv E_2$. The representation is in terms of bases in a compact topological space..
- The bases form a dense G_δ set.
- Any theory of modeloids has either continuum or at most countably many bases.
- An algebraic condition implying at most countably many modeloids.
- Isomorphism classes of modeloids are described in terms of orbits of the topological space and automorphisms of the algebraic representation of elementary equivalence.
- In a nutshell, the results of this paper regarding the number of isomorphism types of a complete finitary theory of modeloids are:
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#iso types	#automorphisms	
#bases	$\leq \omega$	2^ω
$\leq \omega$	**$\leq \omega$**	**$\leq \omega$**
2^ω	**(2^ω)**	**?**

That there countably many or continuum automorphisms of a countable structure is a theorem due to Kueker. The entry (2^ω) indicates that additional assumption is needed.

The structure of the paper:

The first section of the paper gives just the definitions and statements of theorems and propositions. The proofs are given in the next section. If interested in a proof follow the link and return. (I think this draft is self-contained if you take the references for granted; I can send a .pdf of some of them).

Notation:

Modeloids are defined in [2] as equivalence relations between finite non-repeating words from a set A. We will be dealing with countably infinite sets so we take A as N, the set of natural numbers and use N" for these words (" resembles $^{1-1}$); the original paper used "N-hat" for this set.

A word **w** in N" is a one-to-one function from {1, …, k} to N or the empty word; L_k(N") is the set of these words. We sometimes write this as $\langle \mathbf{w}(i) \mid i \le k \rangle$. An initial segment of **w** of length $j \le k$ is the word **w**[j] $= \langle \mathbf{w}(i) \mid i \le j \rangle$. If **w** is a word of length k and $x \in N$ does not appear in **w** then **w**x is the word of length k+1 with **w**x(k+1) = x; if x appears in **w** then **w**x = **w.** The words **w** for which **w**(i) = i are identities, denoted as **id**'s; for $k \in N$, the identity on k is $\mathbf{id_k}$.

N! is the set of all permutations of N and for $k \in N$, k! is the set of all permutations of {1 …, k}. Permutations $p \in k!$ act on word w **w** of length k and the result is the word $\mathbf{w}p = \langle \mathbf{w}(p(i)) \mid i \le k \rangle$

Definition 1: Modeloids as a first order logic theory

The language has relations $E_1, E_2, \ldots, E_k, \ldots$, where each E_k is a 2k-ary relation $E_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ and the axioms of the theory of modeloids, ThM, are:

1. $E_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ implies $(x_i \neq x_j)$ and $(y_i \neq y_j)$ for any $i \neq j$
2. $E_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ is an equivalence relation between the tuples of variables $\mathbf{x} = (x_1, \ldots, x_k)$and $\mathbf{y} = (y_1, \ldots, y_k)$. That is: $E(\mathbf{x}, \mathbf{x})$; $E(\mathbf{x}, \mathbf{y})$ implies $E(\mathbf{y}, \mathbf{x})$; and [$E(\mathbf{x}, \mathbf{y})$ and $E(\mathbf{y}, \mathbf{z})$ implies $E(\mathbf{x}, \mathbf{z})$].
3. $E_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ implies $E_k(x_{p(1)}, \ldots, x_{p(k)}, y_{p(1)}, \ldots, y_{p(k)})$ for any k and any permutation $p \in k!$ (these axioms are for individual k's and p's)
4. If $k \leq n$ and $E_n(x_1, \ldots, x_n, y_1, \ldots, y_n)$ then $E_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$

These axioms are first order expressions of the definition of modeloids given in [2] on page 50. All results of model theory apply to ThM. A model of ThM is a modeloid, essentially an equivalence relation E on N"; we will use E to represent any of the E_k's and and then $E_1, E_2, \ldots$ to denote different modeloids (strictly speaking, it should be something like $E_{k,1}, E_{k,2}, \ldots$).

Definition: A finitary modeloid is one where each E_k has only finitely many equivalence classes. This can be expressed in first order for individual k's.

Theorem 1: If Vaught's Conjecture is true for theories of finitary modeloids then it is true for all first order theories in finite languages.

Definition 2: (outlines of modeloids): (see Definition 3.1 and Propositions 3.2 and 3.4 of [2])

For any equivalence relation E on N" it is natural to consider the set of its equivalence classes, N"/E. For a modeloid E on N", N"/E has the following properties:

1. The partial order ≤ on N"/E defined below is a tree:

   v/E ≤ **w**/E iff length(**v**) ≤ length(**w**) and [**v**E(an initial segment of **w**)].

2. The action of p ϵ k! on L_k(N"/E) defined by (**w/**E)p **= (w**p/E) is well defined.
3. If permutations p and q in n! agree up to k ≤ n then (**w**p)[k]/E = (**w**q)[k]/E.
4. For any **u**, **v** in N" there is a **w** in N" and a permutation p such that **u**/E ≤ **w**/E and **v/E** ≤ (**w**p)/E.

(In (4) we could take **w** as the word **u** followed by **v** (removing repetitions) and permute **w** so that its initial segment is **v**).

The tree T(E) = N"/E (with ≤ and actions) is called the outline of E.

Definition 3 (Outline in general): A general outline is a fist order structure (T, ≤, … , p, …) where p's are symbols for permutations from k! for k ϵ N

It satisfies the following conditions corresponding to the above:

1. A general outline is a tree (T , ≤), very much like (N"/E, ≤) above, in which each t ϵ T has finitely many predecessors. Level k of the tree, L_k(T), consists of nodes of the tree with k + 1 predecessors (counting the empty word). When j ≤ k and t ϵ L_k(T) then t[j] is the unique predecessor of t in L_j(T).
2. The outline also has permutations from k! acting on L_k(T): the result of an action by p ϵ k! on t ϵ L_k(T) is in L_k(T) and is denoted by tp.

We require the standard properties of actions of the groups k! on L_k(T) plus the properties corresponding to the properties of N"/E (3) and (4) above:

3. For t ϵ L_n(T), if permutations p and q in n! agree up to k ≤ n then (tp)[k] = (tq)[k].
4. For any r, s in T, there is a tϵ T and a permutation p such that r ≤ t and s ≤ tp.

Definition 4 (branches): A branch b of an outline T is a maximal linearly ordered set of T. (In symbols, b = { b_k ϵ T| b_k ϵ L_k(T) & k ϵ N} and b_k < b_{k+1})

Definition 5 (bases of an outline): A basis of an outline T is a branch b of T such that

T = { t ϵ T | t ≤ b_kp for some k and p ϵ k! }.

Speaking informally, actions on b cover T. In the specific case of N"/E, the branch b = { $\mathbf{id_k}$/E | k ϵ N } is a basis of N"/E.

Theorem 2: Outlines with finite levels have bases.

Follows from Konig's Lemma and Property (4) of outlines.

Definition 6: (modeloid of a basis)

Given a basis b = {b_k | k ϵ N} of T we define its modeloid E(b) on N" from the map E:N" -> T:

For **w** in L_k(N") E(**w**) = (b_mp)[k] ,where p ϵ m! is such that p(i) = **w**(i) for i . ≤ k. The equivalence relation is then defined as

vE**w** iff E(**v**) = E(**w**).

All that precedes is well defined. This modeloid is referred to as E(b)

Theorem 3: Given an outline T and its basis b, the outline of the modeloid E(b) is isomorphic to T.

Definition 7: (the last level of the outline) The set of all branches of an outline T is called the last level of T, denoted as NT. It has a natural topology with permutations of N! acting on NT (see Construction 3.9 of [2]).

Proposition 3.17 of [2] states that the last level is compact iff all levels of the outline are finite.

Theorem 4 The set of bases of an outline T is a dense G_δ subset of NT. Action by p from N! on a basis is a basis.

The key operation on modeloids is the derivative:

Definition 8: The **derivative** of modeloid E is the relation E' on N"defined as follows:

vE'**w** iff **v**E**w** & **[(**$\forall$x)($\exists$y)[**v**xE**w**y] & ($\forall$y)($\exists$x)[**v**xE**w**y]]

It is easy to check that if E is a modeloid then so is E', a sub-modeloid of E (see propositions 2.2 and 2.3 of [2]). The derivative is inspired by Ehrenfeucht-Fraisse games (see [3]).

In terms of outlines we have

vE'**w** iff {**v**x/E | x ϵ N} = {**w**y/E | y ϵ N}.

The derivative can be iterated; for limit ordinals we take the intersection of the previous derivatives. This process ends at a countable ordinal β with a modeloid E^β which is its own derivative. We call the first such β the complexity of the modeloid. :

Definition 8: A modeloid E is basic if E' = E.

A basic modeloid E is, essentially, a closed subgroup H of N! (see Proposition 5.9 of [2]):

vE**w** iff for some h ϵ H, $\mathbf{wv^{-1}}$ is included in h. (somewhat familiar expression)

??? Theorem 4: Any complete first-order theory Th (in a finite language) whose modeloid theory ThM is basic, is ω-categorical. **(see …)**

Definition 9: The derivative of a modeloid E, E', also has an outline and there is a natural projection π of T(E) onto T(E')defined by π(**w**/E') = **w**/E.

Generalizing this to abstract outlines:

Definition 10: A map π from an outline T' onto an outline T is a valid projection of outlines if the following conditions hold:

1. π maps $L_k(T')$ onto $L_k(T)$
2. If s < t then π(s) < π(t)
3. For $t \in L_k(T')$ and $p \in k!$ π(tp) = π(t)p
4. If t_1, t_2 are $L_k(T')$ and $\pi(\{ t \in L_{k+1}(T) \mid t_1 < t \} = \pi(\{ t \in L_{k+1}(T) \mid t_2 < t \}$ then $t_1 = t_2$.

Theorem 5: If b' is a basis in T' then π(b') is a basis in T and E(π(b'))' = E(b'). (That is the derivative of the projection is the original modeloid of T').

The step from T to T' can be iterated similarly to the iteration of the derivative; that is

Definition 11: ($T^{<\omega}$):

$T^{<\omega} = T \leftarrow T^1 \leftarrow \ldots \leftarrow T^n \leftarrow \ldots$ with projections $\pi_{n+1}: T^{n+1} \to T^n$ satisfying the conditions of Definition 10. We will write π for these projections.

For a modeloid E on N" we write $T^{<\omega}(E)$ for the sequence $(N''/E) \leftarrow (N''/E') \leftarrow \ldots \leftarrow (N''/E^n) \leftarrow \ldots$ with the projections being $\mathbf{w}/E^{n+1} \to \mathbf{w}/E^n$. The reason for using $T^{<\omega}$ is that T^ω should be used for the outline of E^ω (the intersection of E^n 's)

Theorem 6: For modeloids E_1 and E_2,

$$E_1 \equiv E_2 \text{ iff } T^{<\omega}(E_1) \approx T^{<\omega}(E_2).$$

In words: E_1 is elementarily equivalent to E_2 iff $T^{<\omega}(E_1)$ is isomorphic to $T^{<\omega}(E_2)$.

In other words: a complete theory of modeloids corresponds to an outline $T^{<\omega}$

This is the "algebraic" representation of ≡ between modeloids I mentioned at the beginning.

Definitions 4, 5, and 6 can be extended to the "outline" $T^{<\omega}$:

Definition 12: (branches): A branch b of an outline $T^{<\omega}$ is $b = \{ b_n \mid n \in N \text{ and } b_n \in L_n(T^n)\}$ such that $b_n < \pi(b_{n+1})$ where π is the projection from T^{n+1} onto T^n.

Note: In general, one could consider branches of $T^{<\omega}$ as $b_{k,n} \in L_k(T^n)$ with appropriate constraints on the relationships but the above simplifies the presentation.

Definition 13 (bases of an outline): A basis of an outline $T^{<\omega}$ is a branch b of $T^{<\omega}$ such that

$$T^{<\omega} = \{ t \in T^{<\omega} \mid t \leq \pi(b_n p) \text{ for some n and } p \in n! \}.$$

Note: If $t \in L_k(T^n)$ with k > n then we need to use b_k and project $b_k p$ onto T^n .

Definition 14: Given a basis $b = \{b_n \mid n \in N\}$ of $T^{<\omega}$ we define its modeloid E(b) on N" from the projection of the basis b: $\{\pi(b_n) \mid n \in N \}$. (see Definition 6).

All above is well-defined.

Theorem 7: Given a basis b for an outline $T^{<\omega}$ the outline $T^{<\omega}(E(b))$ of the modeloid E(b) is isomorphic to $T^{<\omega}$.

Theorem 8: The isomorphism types of a modeloid theory ThM are the isomorphism types of modeloids E(b) where b is a basis of $T^{<\omega}$.

This is a corollary of the previous theorems: If ThM is a complete theory of modeloids and $T^{<\omega}$ its corresponding outline (see Theorem 6) then any countable model E of ThM is isomorphic to E(b) for some basis b of $T^{<\omega}$

Theorem 9: The outline $T^{<\omega}$ has $\leq \omega$ or 2^{ω} bases. (The same holds for any other outline).

Theorem 10: Given bases b_1 and b_2 of $T^{<\omega}$, their modeloids

$E(b_1)$ and $E(b_2)$ are isomorphic iff for some automorphism A of T and some $p \in N!$, $b_2 = A(b_1\ p)$.

In other words: two modeloids of ThM defined by their bases are isomorphic iff one basis is an automorphic image of a basis from an orbit of the other basis.

Definition of sticks …

Another representation of modeloids:

As stated in the original paper modeloids can also be defined as “sets of partial automorphisms”; page 49 of [2]. To describe the correspondence we define, for words **v** and **w** from N” of the same length,

$\mathbf{wv}^{-1}$ is the map which sends **v(**i) to **w**(i).

Then, given a modeloid as an equivalence relation E, define

$M_E = \{(\mathbf{v}\text{->}\mathbf{w}) \mid \mathbf{v}E\mathbf{w}\}$

M_E is an inverse semigroup, as pointed out by Dana Scott, with some additional properties.

Given such inverse semigroup M we can then define, in turn, an equivalence relation E_M

$\mathbf{v}E_M\,\mathbf{w}$ iff the map $\mathbf{wv}^{-1}$ belongs to M. (akin to congruences in group theory).

Paper [3] analyzes modeloids in this setting in more details and generalizes it to categorical modeloids.

A typical example of a basic modeloid in terms of bijections is

M_H = {m | 👓h ϵ H) [m is included in h]}. T

See Proposition 5.9 of [2].

References:

[1] R. Vaught, "Denumerable models of complete theories", Infinitistic Methods (Proc. Symp. Foundations Math., Warsaw, 1959) Warsaw/Pergamon Press (1961) pp. 303–321

[2] M. Benda, Modeloids I, Transactions of the American Mathematica Society, Vol. 250, , pp. 47-90.

[3] Ehrenfeucht-Fraisse games (see references in Wikipedia:

https://en.wikipedia.org/wiki/Ehrenfeucht–Fraïssé_game

[4] L. Tiemens, D.S.Scott, C. Benzmuller, M. Benda, Computer-supported Exploration of Categorical Axiomatization of Modeloids.

Symbols

≤ ≠ ω ✂. 👓. ^ . β ≡ ≥ ⊢ ∩ ← →

⊊ ∪ ≃ ⊨ ⋀ ⋁ ⋂ ⋃ ± ∑ ∏